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The Steinberg Module and the Cohomology of Arithmetic Groups

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INTRODUCTION

Let $\mathbf{G}$ be a connected algebraic $\mathbb{Q}$-group, $\mathbf{S}_G$ the Steinberg representation of $G = \mathbf{G}(\mathbb{Q})$. Recall that $\mathbf{S}_G$ may be realized on the reduced integral homology of the Tits building of parabolic $\mathbb{Q}$-subgroups of G (see Section 1). In this article, we combine facts about $\mathbf{S}_G$ with the Borel–Serre Duality Theorem (see (3.1)) and basic Lie theory to derive new results on the cohomology of arithmetic subgroups Γ of G. These are summarized below. Our sharpest results apply to $H^v(\Gamma, -)$ (v is the virtual cohomological dimension of Γ) where $\mathbf{G}$ is split over $\mathbb{Z}$. Some of our work generalizes that of Ash, Ash and Rudolph, and Lee and Szczarba on $SL_n(\mathbb{Z})$. See [A1, A-R, L-S, L-S1].

In earlier papers on this topic, an important ingredient has been a simplicial complex Y of dimension equal to vcd $SL_n(\mathbb{Z})$ on which $SL_n(\mathbb{Z})$ acts cocompactly with finite cell stabilizers. The existence of such a Y for a general Chevalley group has not been verified. (There has been recent success with $SP_4(\mathbb{Z})$ in [M-M].) Roughly speaking, we avoid this issue by using $\mathbf{S}_G$ as a substitute for the top chain group of Y.

Assume for now the $\mathbf{G}$ is semisimple, split over $\mathbb{Z}$, and has no factor of type A_1.

THEOREM 1 (see (3.5)). *Let p be an odd prime, K an algebraic closure of $\mathbb{F}_p$. Let V be an irreducible rational $G(K)$ module with a highest weight which is even, positive, and less than p on each of the simple coroots. Then there is a nonzero $G(\mathbb{F}_p)$ homomorphism $H^v(\Gamma(p), \mathbb{Z}) \to V$. Here $\Gamma(p) = \ker G(\mathbb{Z}) \to F(\mathbb{F}_p)$ is the full congruence subgroup of level p.*

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This result follows from the Duality Theorem and the fact that $G(\mathbb{Z})$ acts transitively on the Borel subgroups of G. In [L-S] it was proved for $\mathbf{G} = \mathbf{SL}_n$ that

$$H^{\nu}(\Gamma(3), \mathbb{Z}) \simeq \mathbf{S}_G(\mathbb{F}_3),$$

the Steinberg module of $SL_n(\mathbb{F}_3)$. Recall that the highest weight of $\mathbf{S}_G(\mathbb{F}_p) \otimes K$ is $p-1$ on each simple coroot, so when $p = 3$ this is the only $G(K)$ module satisfying the hypotheses of Theorem 1.

Let $\mathbf{T}$ be a maximal $\mathbb{Q}$-split torus of $\mathbf{G}$, $\mathbf{B} = \mathbf{TU}$ a Borel subgroup of $\mathbf{G}$. $W = \mathrm{Norm}_G(T)/T$. It is standard (cf. [B-S]) that there is an element $\tau \in \mathbf{S}_G$ such that

(i) $\tau \cdot t = \tau$ for all $t \in T$

(ii) $\tau \cdot w = \varepsilon(w)\tau$ for $w \in W$ (ε is the sign character of W)

(iii) τ freely generates $\mathbf{S}_G$ as a $\mathbb{Z}U$ module.

THEOREM 2 (see (2.3)). *Assume* $G = \mathbf{SL}_n$. *Then* $\mathbf{S}_G$ *is generated by* τ *over* $\mathbb{Z}[G(\mathbb{Z})]$.

This result is proven (though stated differently) in [L-S, A-R]. It has several consequences. For example, if k is a field in which $(n+1)!$ is invertible, we get an injection

$$H_\nu(SL_n(\mathbb{Z}), A) \hookrightarrow \{a \in A \mid a \cdot t = a \text{ for } t \in T(\mathbb{Z}) \text{ and } a \cdot w = \varepsilon(w)a \text{ for } w \in W\}$$

for every $kSL_n(\mathbb{Z})$ module A. From this we get

COROLLARY 1 (see (3.10), (3.11)). (1) $H_\nu(SL_n(\mathbb{Z}), \mathbb{Z})$ *is finite and its torsion primes divide* $(n+1)!$.

(2) $H_\nu(SL_n(\mathbb{Z}), V) = 0$ *for any constituent* V *of the exterior algebra on* $E \oplus E$ *where* E *is the natural representation of* $SL_n(\mathbb{C})$ *on* $\mathbb{C}^n$. *In particular,* $H_\nu(SL_n(\mathbb{Z}), \mathrm{Ad}) = 0$, *where* Ad *is the adjoint representation.*

COROLLARY 2 (see (3.6), (3.8)). (1) $H_{\nu-1}(SL_n(\mathbb{Z}), \mathbb{Z})$ *has no p-torsion, p a prime, for* $p > n+1$.

Let $\mathscr{A} = \mathrm{Ann}_{kSL_n(\mathbb{Z})}\tau$, *and set* $\hat{\mathscr{A}} = \{\Sigma c_\gamma \gamma \mid \Sigma c_\gamma \gamma^{-1} \in \mathscr{A}\}$. *Then*

(2) $H^{\nu-1}(SL_n(\mathbb{Z}), k) \simeq \ker \deg\colon H_0(G(\mathbb{Z}), \mathscr{A}) \to k$.

(3) $H^{\nu-1}(SL_n(\mathbb{Z}), \mathbf{S}_G \otimes k) \simeq \mathscr{A} \cdot \hat{\mathscr{A}} \backslash \mathscr{A} \cap \hat{\mathscr{A}}$.

The $SL_n(\mathbb{Z})$ module $\mathscr{A}$, being the first syzygy of $\mathbf{S}_G$, deserves a closer look. In [R1], it is shown how certain elements in $\mathscr{A}$ are related to "nonminimal modular symbols."

In an earlier version of this paper, similar estimates for the torsion in $H_1(G(\mathbb{Z}), \mathbb{Z})$, for an arbitrary Chevalley group, were obtained from a vanishing theorem for $\mathrm{Ext}^{\nu-1}_{G(\mathbb{Z})}(\mathbf{S}_G, k)$. However, Steinberg has shown me that the exact structure of $H_1(G(\mathbb{Z}), \mathbb{Z})$ can be computed using commutator relations. Steinberg communicated the following result, gave a proof, and suggested these be included here. His proof was alluded to in [B1], where parts of the theorem may be found. Any indelicacy in the treatment is due to the present author.

THEOREM 3 (see (3.12)). *Let* $\mathbf{G}$ *be a Chevalley group over* $\mathbb{Z}$ *with irreducible root system.*

(1) *If* $\mathbf{G}$ *is not of type* A_1, B_2, *or* G_2, *then* $H_1(G(\mathbb{Z}), \mathbb{Z}) \simeq F/2F$, *where* F *is the fundamental group of* $\mathbf{G}$.

(2) *If* $\mathbf{G}$ *has type* G_2, *then* $H_1(G(\mathbb{Z}), \mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$.

(3) *If* $\mathbf{G}$ *has type* B_2, *then* $H_1(G(\mathbb{Z}), \mathbb{Z}) \simeq F \otimes \mathbb{Z}/2\mathbb{Z}$. (*Note that* F *is either trivial or of order* 2.)

(4) *If* $\mathbf{G}$ *has type* A_1, *then* $H_1(G(\mathbb{Z}), \mathbb{Z}) \simeq \mathbb{Z}/12\mathbb{Z}$ *or* $(\mathbb{Z}/2\mathbb{Z})^2$, *according to whether* $\mathbf{G}$ *is* $\mathbf{SL}_2$ *or* $\mathbf{PSL}_2$.

For the remainder of this introduction, we drop the assumption that $\mathbf{G}$ is split over $\mathbb{Q}$. Let $\mathbf{P}$ be a parabolic $\mathbb{Q}$-subgroup of $\mathbf{G}$.

The original goal of this work was to describe the restriction map

$$H^\nu(\Gamma, A) \to H^\nu(\Gamma \cap P, A)$$

in terms of "modular symbols" (cf. [A1, A-R]). In the present viewpoint, a (universal minimal) modular symbol is nothing but an element of $\mathbf{S}_G$. The minimal modular symbols are known to generate $H^\nu(\Gamma, A)$. The proof of this fact boils down, via duality, to the tautology that $\mathbf{S}_G \otimes A \to \mathbf{S}_G \otimes_\Gamma A$ is surjective. Since $\Gamma \cap P$ is also a duality group with the same dualizing dimension, one might hope to describe the restriction map in terms of $\mathbf{S}_G$ and $\mathbf{S}_P$. This leads one to focus on the G module structure of $\mathbf{S}_G$, and its restriction to P in particular. Write $\mathbf{P} = \mathbf{LN}$ (Levi decomposition), and set

$$[\mathbf{S}_G, N] = \langle x - x \cdot n \mid x \in \mathbf{S}_G, n \in N \rangle.$$

There is an exact sequence of P modules

$$0 \longrightarrow [\mathbf{S}_G, N] \longrightarrow \mathbf{S}_G \xrightarrow{s_P} \mathbf{S}_L \longrightarrow 0$$

and an L homomorphism $\sigma_L \colon \mathbf{S}_L \to \mathbf{S}_G$ which is a section for s_P. We use s_P to define a transfer map

$$\Theta_{P,A} \colon H_*(\Gamma, \mathbf{S}_G \otimes A) \to H_*(\Gamma \cap P, \mathbf{S}_L \otimes A)$$

which is well defined even though $[\Gamma : \Gamma \cap P] = \infty$. This transfer corresponds to the restriction map

$$H^{\nu-*}(\Gamma, A) \to H^{\nu-*}(\Gamma \cap P, A)$$

under duality. When $* = 0$, we give a more explicit formula for $\Theta_{P,A}$ involving a sum over certain Weyl group elements. Geometrically speaking, the explicit formula for $\Theta_{P,A}$ expresses the boundary of a big modular symbol as a sum of smaller modular symbols, taking into account Γ conjugacy and the twisting by A.

When $\mathbf{G} = \mathbf{SL}_3$, this leads to an isomorphism between the cuspidal cohomology of $\Gamma < SL_3(\mathbb{Z})$ and a certain space of functions $f: SL_3(\mathbb{Z})/\Gamma \to A$. This is not described here (see [R]). In keeping with the somewhat general theme of this paper, we content ourselves with the following application of the formula for $\Theta_{P,A}$. The result can be obtained using Eisenstein series, but perhaps the algebraic nature of our proof is of some interest.

Let V be a rational $G(\mathbb{C})$ module, viewed as a $G(\mathbb{Z})$ module via the inclusion $G(\mathbb{Z}) \subsetneq G(\mathbb{C})$. Let Δ and Δ_L be simple $\mathbb{Q}$-roots for $\mathbf{G}$ and $\mathbf{L}$, respectively. Let λ be the *lowest* $\mathbb{Q}$-weight of V with respect to Δ. Write $\lambda = \sum_{\alpha \in \Delta} c_\alpha \lambda_\alpha$, where the λ_α's are the fundamental $\mathbb{Q}$-weights.

THEOREM 4 (see (4.12)). *Assume Γ is normal in $G(\mathbb{Z})$ with finite index. If $c_\alpha < 0$ for all $\alpha \in \Delta \backslash \Delta_L$, then the restriction map $H^\nu(\Gamma, V) \to H^\nu(\Gamma \cap P, V)$ is surjective.*

The contents of this paper are arranged as follows. In Section 1, we define the Steinberg module $\mathbf{S}_G$ for any connected F-group $\mathbf{G}$ and study the restriction to $\mathbf{S}_G$ to parabolic F-subgroups of G. The result here (Proposition (1.1)) is well known (cf. [Ro]) when F is finite or p-adic. In Section 2, we assume G is reductive and split over F. This enables us to describe the action of $G(\mathbb{Q})$ more explicitly. Similar computations can be found in [S1], where the finiteness of the base field is not really used. We also discuss the cyclicity of $\mathbf{S}_G$ for $SL_n(\mathbb{Z})$.

From Section 3 on, we take $F = \mathbb{Q}$ and apply our results to group cohomology, invoking the duality theorem (3.1) at every turn. Section 4 is devoted to the restriction map and the proof of Theorem 4.

I thank my thesis advisor, Avner Ash, for his encouragement. Armand Borel read an earlier version of this work and corrected my misconceptions at several points. I especially thank Robert Steinberg, who, in addition to supplying Theorem 3, made numerous suggestions for improvement and pointed me to the very relevant papers [B1, S1], which I had overlooked.

1. The Steinberg Representation

Fix an arbitrary field F. For any connected algebraic group $\mathbf{G}$ defined over F, set $G = \mathbf{G}(F)$. If R is a proper subring of F and $\mathbf{G}$ is defined over R, we write $G(R)$ instead of $\mathbf{G}(R)$. Let $l = F$-rank of $\mathbf{G}$. Recall the definition of the Tits building $\mathscr{T}_G$ of G [B-S]. This is an $l-1$ dimensional simplicial complex whose vertices correspond to the maximal parabolic F-subgroups $\mathbf{P}$ of $\mathbf{G}$. If $0 \leqslant j \leqslant l-1$, the $l-j$ simplicies of $\mathscr{T}_G$ are the parabolic F-subgroups of $\mathbf{P}$ such that the F-rank of the derived group of $\mathbf{G}/R_u\mathbf{P}$ is $j-1$. ($R_u\mathbf{P}$ is the unipotent radical of $\mathbf{P}$.) $\mathscr{T}_G$ has the homotopy type of a bouquet of $l-1$ spheres. G acts on $\mathscr{T}_G$ on the right.

Define

$$\mathbf{S}_G = \tilde{H}_{l-1}(\mathscr{T}_G, \mathbb{Z}).$$

This is a module for G, called the "Steinberg representation" of G. Note that $\mathbf{S}_G$ is a free abelian group.

Fix a minimal F-parabolic $\mathbf{B}$ of $\mathbf{G}$. Note that $\mathbf{S}_G$ is a submodule of $C_{l-1}(\mathscr{T}_G, \mathbb{Z}) \simeq \mathbb{Z} \otimes_{\mathbb{Z}B} \mathbb{Z}G$. In fact,

$$\mathbf{S}_G \simeq \bigcap_P \ker[\mathbb{Z} \otimes_{\mathbb{Z}B} \mathbb{Z}G \to \mathbb{Z} \otimes_{\mathbb{Z}P} \mathbb{Z}G],$$

where $\mathbf{P}$ runs over the parabolic F-subgroups of $\mathbf{G}$ which are minimal with respect to properly containing $\mathbf{P}$. The right-hand side is our preferred model for $\mathbf{S}_G$.

If $\mathbf{T}$ is a maximal F-split torus of $\mathbf{B}$, then the "apartment" corresponding to $\mathbf{T}$ is

$$\tau = \sum_{w \in W} \varepsilon(w) \otimes w \in \mathbf{S}_G,$$

where $W = \mathrm{Norm}_G(T)/\mathrm{Cent}_G(T)$ is the relative Weyl group of T in G, and ε is its sign character. This is well-defined because $\mathrm{Cent}_G(T) \leqslant B$. If $\mathbf{U}$ is the unipotent radical of $\mathbf{B}$, then τ freely generates $\mathbf{S}_G$ as a $\mathbb{Z}U$-module.

Let $\mathbf{P}$ be a parabolic F-subgroup of $\mathbf{G}$, $\mathbf{L}$ a Levi subgroup of $\mathbf{P}$. The projection $\pi\colon \mathbf{P} \to \mathbf{L}$ induces a canonical isomorphism $\mathbf{S}_P \simeq \mathbf{S}_L$, and we identify these two spaces. The action of P on $\mathbf{S}_P$ is the pullback of the L-action. Let

$$s_P\colon \mathbb{Z} \otimes_{\mathbb{Z}B} \mathbb{Z}G \to \mathbb{Z} \otimes_{\mathbb{Z}B} \mathbb{Z}P$$

be the projection: $s_P(1 \otimes g) = 1 \otimes g$ if $g \in P$, zero otherwise.

Set $\mathbf{B}_L = \pi(\mathbf{B})$, a minimal F-parabolic of $\mathbf{L}$. We define a map

$$\sigma_L\colon \mathbb{Z} \otimes_{\mathbb{Z}B_L} \mathbb{Z}P \to \mathbb{Z} \otimes_{\mathbb{Z}B} \mathbb{Z}G$$

as follows. Fix a maximal F-split torus $\mathbf{T}$ of $\mathbf{L}$ (hence of $\mathbf{G}$) contained in $\mathbf{B}$. Let

$$W^P = \{w \in W \mid wB_L w^{-1} \subseteq B\}$$

be the distinguished representatives for W/W_P, where $W_P = \mathrm{Norm}_L(T)/\mathrm{Cent}_L(T)$. We set

$$\sigma_L(1 \otimes p) = \sum_{w \in W^P} \varepsilon(w) \otimes wp.$$

We may view σ_L as a map

$$\sigma_L : \mathbf{S}_L \to \mathbb{Z} \otimes_{\mathbb{Z}B} \mathbb{Z}G$$

via the inclusions $\mathbf{S}_L \subsetneq \mathbb{Z} \otimes_{\mathbb{Z}B_L} \mathbb{Z}L \subsetneq \mathbb{Z} \otimes_{\mathbb{Z}B_L} \mathbb{Z}P$.

Let $\mathbf{U}$, $\mathbf{U}_L$, and $\mathbf{N}$ be the unipotent radicals of $\mathbf{B}$, $\mathbf{B}_L$, and $\mathbf{P}$, respectively. We have $\mathbf{B} = \mathbf{N}\mathbf{B}_L$. Set

$$[\mathbf{S}_G, N] := \mathbb{Z}\langle x \cdot (n-1) \mid x \in \mathbf{S}_G, n \in N \rangle.$$

Put $\tau = \sum_W \varepsilon(w) \otimes w$ as above and analogously let

$$\tau_L = \sum_{W_P} \varepsilon(w) \otimes w \in \mathbf{S}_L.$$

(1.1) PROPOSITION. (1) *$s_P(\tau) = \tau_L$ and $\sigma_L(\tau_L) = \tau$. Hence $s_P(\mathbf{S}_G) = \mathbf{S}_L$ and $\sigma_L(\mathbf{S}_L) \subseteq \mathbf{S}_G$.*

(2) *The sequence*

$$0 \longrightarrow [\mathbf{S}_G, N] \longrightarrow \mathbf{S}_G \xrightarrow{s_P} \mathbf{S}_L \longrightarrow 0$$

is an exact sequence of P modules, and σ_L is an L equivariant section of s_P.

(3) *The map $(x \otimes p) \mapsto \sigma_L(x)\, p : \mathbf{S}_L \otimes_{\mathbb{Z}L} \mathbb{Z}P \to \mathbf{S}_G$ is an isomorphism of P modules.*

Proof. Part (1) follows from the fact that

$$s_P(1 \otimes w) = 1 \otimes w \text{ if } w \in W_P \text{ and is zero otherwise.}$$

It is clear from the definitions that s_P and σ_L are P and L equivariant, respectively. Also, if $l \in L$,

$$s_P \sigma_L(1 \otimes l) = s_P\left(\sum_{w \in W^P} \varepsilon(w) \otimes wl\right) = 1 \otimes l$$

because $w \in W^P$ and $wl \in P$ together imply $w = 1$. Now $\mathbf{S}_G$ and $\mathbf{S}_L$ have bases $\{\tau \cdot u \mid u \in U\}$ and $\{\tau_L \cdot u \mid u \in U_L\}$, respectively. Writing $u \in U = NU_L$ as $u = nu_L$, we have

$$\tau \cdot u = \tau \cdot u_L(n-1) + \tau \cdot u_L = \tau \cdot u_L(n-1) + \sigma_L(\tau_L \cdot u_L)$$

so

$$\mathbf{S}_G = [\mathbf{S}_G, N] + \sigma_L(\mathbf{S}_L).$$

It follows that $\ker s_P \subseteq [\mathbf{S}_G, N]$. This finishes (2).

For part (3), we need only observe that $\mathbf{S}_L \otimes_{\mathbb{Z}L} \mathbb{Z}P$ has a basis

$$\{\tau_L \cdot u_L \otimes n \mid u_L \in U_L, n \in N\}. \quad \blacksquare$$

(1.2) *Remark.* Suppose $Q = \mu^{-1} P \mu$, for some $\mu \in G$. Then we have an isomorphism

$$c_\mu \colon \mathbb{Z} \otimes_{\mathbb{Z}[\mu^{-1}B\mu]} \mathbb{Z}Q \to \mathbb{Z} \otimes_{\mathbb{Z}B} \mathbb{Z}P$$

which maps $\mathbf{S}_P \to \mathbf{S}_Q$. Note that $c_\mu(\tau_P) = \tau_Q$, where τ_Q corresponds to $\mu T \mu^{-1}$ and for $q \in Q$, $\eta \in \mathbf{S}_Q$, we have

$$c_\mu(\eta \cdot q) = (c_\mu \eta) \cdot \mu q \mu^{-1}.$$

2. Further Results when G is Split

We now assume $\mathbf{G}$ is split over F, and get more explicit information about the G module $\mathbf{S}_G$. First of all, the action on G factors through G modulo its radical, which is a Chevalley group over F. We may (and shall) assume that G itself is a Chevalley group. The basic reference here is [S]. Fix a maximal torus $\mathbf{T}$ contained in a Borel subgroup $\mathbf{B}$ as before. Let Φ^+ and Δ be the corresponding set of positive and simple roots, respectively. For each $\alpha \in \Phi$ we have a homomorphism $\varphi_\alpha \colon SL_2(F) \to G$. Set

$$x_\alpha(t) = \varphi_\alpha \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad t \in F,$$

$$X_\alpha = \{x_\alpha(t) \mid t \in F\},$$

$$n_\alpha = \varphi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and

$$h_\alpha(t) = \varphi_\alpha \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

We first give a formula for the action of $g \in G$ in terms of the basis $\{\tau \cdot u \mid u \in U\}$. Every $u \in U$ can be written uniquely as $u = x_\alpha(t)\, u^\alpha$ for $t \in F$ and $u^\alpha \in \prod_{\beta \in \Phi \setminus \{\alpha\}} X_\beta$.

(2.1) PROPOSITION. *The action of $g \in BwB$ on $\tau \cdot u$, $u \in U$, is given using induction on the length of w, as*

$$(\tau \cdot u)\, hv = \tau \cdot (h^{-1} uhv) \qquad \textit{for} \quad h \in T, v \in U,$$

and

$$(\tau \cdot u)\, vn_\alpha wb = \tau \cdot [x_\alpha(-t^{-1}) - 1][n_\alpha^{-1}(uv)^\alpha\, n_\alpha]\, wb,$$

where $v \in U$, $b \in B$, $w \in W$, $uv = x_\alpha(t)(uv)^\alpha$ as above, and we make the convention that $x_\alpha(-t^{-1}) = 0 \in \mathbb{Z}U$ if $t = 0$. Note that $n_\alpha^{-1}(uv)^\alpha\, n_\alpha \in U$.

Proof. The first equality follows from the fact that T fixes τ. For the second, we have

$$(\tau \cdot u)\, vn_\alpha = \tau \cdot x_\alpha(t)\, n_\alpha n_\alpha^{-1}(uv)^\alpha\, n_\alpha.$$

Let $L(\alpha)$ be the standard Levi factor corresponding to α. Then

$$\tau \cdot x_\alpha(t)\, n_\alpha = \sigma_{L(\alpha)}(\tau_L \cdot x_\alpha(t)\, n_\alpha)$$

so it is enough to do the computation for $SL_2(F)$ in $\mathbf{S}_{SL_2(F)}$. We use the relation

$$\begin{aligned}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}&\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -t^{-1} & 1 \\ 0 & -t \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -t^{-1} \\ 0 & 1 \end{pmatrix}\end{aligned}$$

to see that

$$\begin{aligned}\left[1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 1 \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]&\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \left[1 \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - 1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \\ &\quad + \left[1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 1 \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] \cdot \begin{pmatrix} 1 & -t^{-1} \\ 0 & 1 \end{pmatrix}.\end{aligned}$$

Hence $\tau_L \cdot x_\alpha(t)\, n_\alpha = \tau_L \cdot (x_\alpha(-t^{-1}) - 1)$. Now apply $\sigma_{L(\alpha)}$ to finish the proof. ∎

The Steinberg module may be thought of as an action of $G(F)$ on $\mathbb{Z}U(F)$ which extends the regular representation of $U(F)$. Since the formula in (2.1) does not depend on the field, we have the following

(2.2) COROLLARY. *Let K be a field containing F and $\mathbf{H}$ a connected F-subgroup of $\mathbf{G}$ defined by a root sub-system Φ_H of Φ. Fix compatible systems of positive roots $\Phi_H^+ \subset \Phi^+$ and let $\mathbf{U}_H \subset \mathbf{U}$ be generated by the root groups X_α for $\alpha \in \Phi_H^+$. Then the natural inclusion $\mathbb{Z}U_H(F) \subsetneq \mathbb{Z}U(K)$ is an $H(F)$-equivariant homomorphism of Steinberg modules.*

This gives a generalization of the map σ_L from Section 1.

The following result is the basis of most of our subsequent vanishing theorems for the top cohomology of $SL_n(\mathbb{Z})$ with twisted coefficients.

(2.3) THEOREM [A-R, L-S]. *Let $G = SL_n$. If $\mathcal{O}$ is a Euclidean domain in F then*

$$\mathbf{S}_G = \tau \cdot \mathbb{Z}SL_n(\mathcal{O}).$$

Remarks on Proof. Since the result in the above papers is not phrased this way, we will explain the connection. In the notation of [A-R], for instance, it is shown that every modular symbol $[A]$ is a $\mathbb{Z}$ linear combination of "unimodular symbols," which are those $[A]$'s for $A \in SL_n(\mathcal{O})$. In our notation, $[A]$ is $\tau \cdot A$, and by the Solomon–Tits theorem, the $[A]$'s span $\mathbf{S}_G$.

We can give an elementary proof of (2.3) for $SL_2(\mathbb{Z})$ using (2.1) as follows. Let α be the positive root, and let M denote the $SL_2(\mathbb{Z})$ submodule of $\mathbf{S}_G$ generated by τ. We suppose there exists a $t \in \mathbb{Q}$ such that $\tau \cdot x_\alpha(t)$ does not belong to M. Multiplying by $x_\alpha(c)$ for suitable $c \in \mathbb{Z}$, we may assume $|t| < 1$. Multiplying by n_α gives $\tau \cdot (x_\alpha(-t^{-1}) - 1)$, by (2.1), from which it follows that $\tau \cdot x_\alpha(-t^{-1})$ must not belong to M. The denominator of t^{-1} is strictly smaller in absolute value than of t, so by repeating this process, we will eventually find an integer m such that $\tau \cdot x_\alpha(m) \notin M$, and this is a contradiction.

3. GROUP COHOMOLOGY AND THE STEINBERG MODULE OF $G(\mathbb{Q})$

We begin this section with $\mathbf{G}$ being any connected algebraic $\mathbb{Q}$-group. We consider a torsion-free arithmetic subgroup Γ of G. Let A be a right $\mathbb{Z}\Gamma$ module with added restrictions to be specified along the way. Let d be the dimension of the symmetric space associated to $G(\mathbb{R})$, $l = \operatorname{rank}_{\mathbb{Q}} G$, and set $v = d - l$. Note that if $\operatorname{rank}_{\mathbb{Q}} G = \operatorname{rank}_{\mathbb{R}} G$ then $v = \dim U(\mathbb{R})$ by the

Iwasawa decomposition. Let ω be the orientation character of G on M. For brevity we set

$$\mathbf{S}_G^\omega := \omega \otimes \mathbf{S}_G.$$

The connection between $H^*(\Gamma, A)$ and $\mathbf{S}_G$ is described by the following

(3.1) Duality Theorem (Borel–Serre [B-S]). (1) *$H_\nu(\Gamma, \mathbf{S}_G^\omega) \simeq \mathbb{Z}$. Let e be a generator.*

(2) *Cap product with e gives isomorphisms*

$$\cap e\colon H^*(\Gamma, A) \to \mathrm{Tor}^\Gamma_{\nu-*}(\mathbf{S}_G^\omega, A).$$

(3) *Dually, we have natural isomorphisms*

$$H_*(\Gamma, A) \simeq \mathrm{Ext}_\Gamma^{\nu-*}(\mathbf{S}_G^\omega, A).$$

In particular,

$$H^\nu(\Gamma, A) \simeq \mathbf{S}_G^\omega \otimes_\Gamma A \qquad \textit{and} \qquad H_\nu(\Gamma, A) \simeq \mathrm{Hom}_\Gamma(\mathbf{S}_G^\omega, A).$$

(3.2) *Remark.* If Γ is not torsion-free, then (3.1) still holds as long as Γ has a torsion-free normal subgroup Γ_0 for which $[\Gamma : \Gamma_0]$ is invertible in A. For example, if G is a simple Chevalley group, $\Gamma = G(\mathbb{Z})$, and A is a $k\Gamma$ module with k a field of characteristic equal to at least two plus the Coxeter number of G, we can find a prime r so that char k does not divide $|G(\mathbb{Z}/r)|$. Hence we may take Γ_0 to be the full congruence subgroup of level r. Of course if char $k = 0$, then (3.1) holds for any arithmetic subgroup Γ.

Let k be a field. It is not known to me if $k \otimes \mathbf{S}_G$ is irreducible. However, the Duality Theorem gives a result in this direction.

(3.3) Corollary. *Let k be any field, A and B finite dimensional $k\Gamma$-modules (Γ is torsion-free). The natural map $\varphi \mapsto id \otimes \varphi$ induces an isomorphism*

$$\mathrm{Ext}_\Gamma^*(A, B) \to \mathrm{Ext}_\Gamma^*(\mathbf{S}_G^\omega \otimes A, \mathbf{S}_G^\omega \otimes B)$$

in all dimensions. In particular, $\mathrm{End}_G(k \otimes \mathbf{S}_G) = k \cdot id$.

Proof.

$$\begin{aligned}
&\mathrm{Ext}_\Gamma^*(\mathbf{S}_G^\omega \otimes A, \mathbf{S}_G^\omega \otimes B)\\
&\quad\simeq \mathrm{Ext}_\Gamma^*(\mathbf{S}_G^\omega, \mathbf{S}_G^\omega \otimes B \otimes A^*) \qquad \text{(linear algebra)}\\
&\quad\simeq H_{\nu-*}(\Gamma, \mathbf{S}_G^\omega \otimes B \otimes A^*) \qquad \text{((3.1), part (3))}\\
&\quad\simeq H^*(\Gamma, B \otimes A^*) \qquad \text{((3.1), part (2))}\\
&\quad\simeq \mathrm{Ext}_\Gamma^*(k, B \otimes A^*)\\
&\quad\simeq \mathrm{Ext}_\Gamma^*(A, B). \quad \blacksquare
\end{aligned}$$

For the remainder of this section, assume that G is defined and split over $\mathbb{Z}$, and that $\omega|_\Gamma \equiv 1$. Then $G(\mathbb{Z})$ is transitive on the Borel subgroups of $G(\mathbb{Q})$ (see [B, Sect. 6] or [S, Lemma 43]), so

$$\mathbb{Z} \otimes_{B(\mathbb{Q})} G(\mathbb{Q}) = \mathbb{Z} \otimes_{B(\mathbb{Z})} G(\mathbb{Z}).$$

The idea behind the next few results is that, since $\mathbf{S}_G$ sits inside $\mathbb{Z} \otimes_{B(\mathbb{Z})} G(\mathbb{Z})$ and the (co)homology of $B(\mathbb{Z})$ is, for the most part, computable, we should be able to say something about the (co)homology of $G(\mathbb{Z})$, at least in the extreme dimensions.

Let Γ be a torsion-free normal subgroup of $G(\mathbb{Z})$ with finite index, and set, for any subgroup H of G,

$$H_0 = H \cap G(\mathbb{Z})/H \cap \Gamma.$$

We will show how the duality theorem can be used to determine G_0 composition factors in $H^\nu(\Gamma, \mathbb{Z})$. The trivial coefficient module $\mathbb{Z}$ could be replaced by a nontrivial $G(\mathbb{Z})$-module, but the idea is the same.

(3.4) THEOREM. *Suppose A is a G_0-module so that there is $v \in A^{B_0}$ with*

$$\sum_{w \in W} \varepsilon(w) v \cdot w \neq 0.$$

Then $\mathrm{Hom}_{G_0}(H^\nu(\Gamma, \mathbb{Z}), A)$ *is nonzero.*

Note that by hypothesis, v is invariant under $T(\mathbb{Z})$ and $\mathrm{Norm}_{G(\mathbb{Z})} T/T(\mathbb{Z}) = W$, *so the sum makes sense.*

Proof. We will produce a nonzero element of

$$\mathrm{Hom}_{G_0}(H^\nu(\Gamma, \mathbb{Z}), A) \simeq \mathrm{Hom}_{G_0}(\mathbf{S}_G \otimes_\Gamma \mathbb{Z}, A) \simeq \mathrm{Hom}_{G(\mathbb{Z})}(\mathbf{S}_G, A)$$

by restricting an element of

$$\mathrm{Hom}_{G(\mathbb{Z})}(\mathbb{Z} \otimes_{B(\mathbb{Z})} G(\mathbb{Z}), A)$$

to $\mathbf{S}_G$. We have the Frobenius reciprocity isomorphism

$$A^{B_0} \to \mathrm{Hom}_{G(\mathbb{Z})}(\mathbb{Z} \otimes_{B(\mathbb{Z})} G(\mathbb{Z}), A)$$

given by $v \mapsto (\varphi_v \colon 1 \otimes g \mapsto v \cdot g)$. Note that

$$\begin{aligned} \varphi_v(\tau) &= \sum_{w \in W} \varepsilon(w)\, \varphi_v(1 \otimes w) \\ &= \sum_{w \in W} \varepsilon(w) v \cdot w. \end{aligned}$$

Hence φ_v does not vanish on $\mathbf{S}_G$. ∎

(3.5) *Remarks.* (i) Let K be an algebraically closed field of characteristic $p>2$, $\Gamma=\Gamma(p)$ (the full congruence subgroup of level p), A an irreducible rational $G(K)$ module with highest weight $\lambda=\sum c_\alpha \lambda_\alpha$ (the λ_α's are the fundamental weights) where each c_α is even, positive, and less than p. Then the conditions of (3.4) hold with v a highest weight vector for $B(K)$. This shows that A is a $G(\mathbb{F}_p)$ composition factor of $H^\nu(\Gamma(p), \mathbb{Z})$. In particular, $\mathbf{S}_G(\mathbb{F}_p)$ (the Steinberg module of $G(\mathbb{F}_p)$) is always a composition factor of $H^\nu(\Gamma(p), \mathbb{Z})$ for $p>2$.

(ii) We will see later that those $\varphi \in \mathrm{Hom}_\Gamma(\mathbf{S}_G, A)$ which extend to $\mathbb{Z} \otimes_{B(\mathbb{Q})} G(\mathbb{Q})$ are precisely those homology classes in the image of the homology inclusions

$$\bigoplus_\mu H_\nu(\Gamma \cap B^\mu, A) \to H_\nu(\Gamma, A),$$

where μ runs over a set of representatives for $B(\mathbb{Q})\backslash G(\mathbb{Q})/\Gamma$.

In the following items (3.6) through (3.11), we assume $\mathbf{G}=\mathbf{SL}_n$, so that $\nu=(1/2)\,n(n-1)$. We then give some applications of the duality theorem combined with (2.3), which says that $\mathbf{S}_G=\tau \cdot \mathbb{Z}SL_n(\mathbb{Z})$.

Let k be a field in which $(n+1)!$ is invertible. Let $\mathscr{A}$ be the annihilator in $kSL_n(\mathbb{Z})$ of τ. This is a right ideal in $kSL_n(\mathbb{Z})$, hence is a $kSL_n(\mathbb{Z})$ module. For $r=\sum c_g g \in kSL_n$, define $\hat{r}=\sum c_g g^{-1}$.

(3.6) Proposition. (1) $H_0(SL_n(\mathbb{Z}), \mathbf{S}_G)=0$.

(2) $H_1(SL_n(\mathbb{Z}), k\otimes \mathbf{S}_G) \simeq \ker \deg\colon H_0(SL_n(\mathbb{Z}), \mathscr{A}) \to k$.

(3) *For* $n \geqslant 1$, $H_{n+1}(SL_n(\mathbb{Z}), k\otimes \mathbf{S}_G) \simeq H_n(SL_n(\mathbb{Z}), \mathscr{A})$.

(4) *There is an exact sequence*

$$0 \to H^{\nu-1}(SL_n(\mathbb{Z}), k\otimes \mathbf{S}_G) \to [\mathscr{A}\cdot \hat{\mathscr{A}}\backslash \mathscr{A}]$$
$$\to k\otimes \mathbf{S}_G \to k\otimes [\mathbf{S}_G \otimes_{SL_n(\mathbb{Z})} \mathbf{S}_G] \to 0,$$

where the middle map is given by $a \mapsto \tau\cdot \hat{a}$. *In particular,*

$$H^{\nu-1}(SL_n(\mathbb{Z}), k\otimes \mathbf{S}_G) \simeq \mathscr{A}\cdot \hat{\mathscr{A}}\backslash \mathscr{A} \cap \hat{\mathscr{A}}.$$

(5) $H^{n-1}(SL_n(\mathbb{Z}), k\otimes \mathbf{S}_G) \simeq H^n(\mathscr{A})$ *for* $1 \leqslant n \leqslant \nu-1$. *In particular,* $H^1(SL_n(\mathbb{Z}), \mathscr{A})=0$.

Proof. The first assertion is proven in [L-S]. We give a simple proof using (2.3).

Let E be any field, viewed as a trivial $SL_n(\mathbb{Z})$ module. Then

$$\mathrm{Hom}(H_0(SL_n(\mathbb{Z}), \mathbf{S}_G), E)=\mathrm{Hom}_{SL_n(\mathbb{Z})}(\mathbf{S}_G, E).$$

Let φ belong to the right hand side. For $\alpha \in \Delta$, let

$$g_\alpha = \varphi_\alpha \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Using (2.1), it is easy to see that $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)^2$ annihilates τ_2 in $\mathbf{S}_{SL_2}$. Hence

$$0 = \varphi(0) = \varphi(\tau \cdot (1 + g_\alpha + g_\alpha^2)) = 3\varphi(\tau).$$

Also,

$$0 = \varphi(\tau \cdot (1 + s_\alpha)) = 2\varphi(\tau),$$

so $\varphi(\tau) = 0$. By (2.3), $\varphi \equiv 0$. It follows that $H_0(SL_n(\mathbb{Z}), \mathbf{S}_G)$ must be zero.

We have an exact sequence of $SL_n(\mathbb{Z})$ modules

$$0 \to \mathscr{A} \to kSL_n(\mathbb{Z}) \to \mathbf{S}_G \to 0.$$

Moreover,

$$H_*(SL_n(\mathbb{Z}), kSL_n(\mathbb{Z})) = k$$

if $* = 0$ and is zero otherwise, while

$$H^*(SL_n(\mathbb{Z}), kSL_n(\mathbb{Z})) = k \otimes \mathbf{S}_G$$

if $* = \nu$ and is zero otherwise [B-S, Sect. 11]. From the long exact (co)homology sequences, we get (2), (3), and the first part of (5). The second statement now follows from the fact that $\mathbf{S}_G$ has no $SL_n(\mathbb{Z})$ invariants. (See [F] or simply note that a module induced from a subgroup of infinite index has no invariant subspaces of finite rank.)

For (4), we start with the exact sequence

$$0 \to H^{\nu-1}(SL_n(\mathbb{Z}), k \otimes \mathbf{S}_G) \to H^\nu(SL_n(\mathbb{Z}), \mathscr{A}) \to H^\nu(SL_n(\mathbb{Z}), kSL_n(\mathbb{Z}))$$
$$\to H^\nu(SL_n(\mathbb{Z}), k \otimes \mathbf{S}_G) \to 0$$

and use the duality theorem. We have

$$H^\nu(SL_n(\mathbb{Z}), \mathscr{A}) \simeq (\mathbf{S}_G \otimes \mathscr{A})_{SL_n(\mathbb{Z})} \simeq [(\mathscr{A} \backslash kSL_n(\mathbb{Z})) \otimes \mathscr{A}]_{SL_n(\mathbb{Z})}.$$

This last is isomorphic to $\mathscr{A} \cdot \hat{\mathscr{A}} \backslash \mathscr{A}$ via the map

$$r \otimes a \mapsto a \cdot \hat{r}.$$

If

$$0 \to A \to B \to C \to 0$$

is any exact sequence of $kSL_n(\mathbb{Z})$ modules, the sequence

$$H^v(A) \to H^v(B) \to H^v(C) \to 0$$

becomes the obvious sequence

$$\mathbf{S}_G \otimes_{SL_n(\mathbb{Z})} A \to \mathbf{S}_G \otimes_{SL_n(\mathbb{Z})} B \to \mathbf{S}_G \otimes_{SL_n(\mathbb{Z})} C \to 0.$$

Tracing through the identifications, we get the formula for the map

$$H^v(SL_n(\mathbb{Z}), \mathscr{A}) \to H^v(SL_n(\mathbb{Z}), kSL_n(\mathbb{Z})). \quad \blacksquare$$

(3.7) *Remark.* If $\alpha \in \Delta$, then $1 + s_\alpha$ and $1 + g_\alpha + g_\alpha^2$ belong to $\mathscr{A} \cap \hat{\mathscr{A}}$. In fact, if H is a finite subgroup of $SL_n(\mathbb{Z})$ such that $\tau \cdot \sum_{h \in H} h = 0$, then $\sum_{h \in H} h \in \mathscr{A} \cap \hat{\mathscr{A}}$. However,

$$|H| \sum h = \left(\sum h\right)^2 = \left(\sum h\right)\left(\sum h^{-1}\right) \in \mathscr{A} \cdot \hat{\mathscr{A}}$$

and $\operatorname{char}(k) > n+1 \Rightarrow |H|^{-1} \in k$, so in fact, $\sum h \in \mathscr{A} \cdot \hat{\mathscr{A}}$.

(3.8) Corollary. (1) $H^{v-1}(SL_n(\mathbb{Z}), k) \simeq \ker \deg\colon H_0(SL_n(\mathbb{Z})), \mathscr{A}) \to k$.

(2) $H_{v-2}(SL_n(\mathbb{Z})), \mathscr{A}) = 0$.

Proof. The first result follows from the duality isomorphism

$$H^{v-1}(SL_n(\mathbb{Z}), k) \simeq H_1(SL_n(\mathbb{Z}), k \otimes \mathbf{S}_G).$$

For part (2), use (3.6)(3) and duality to get

$$H_{v-2}(SL_n(\mathbb{Z}), \mathscr{A}) \simeq H_{v-1}(SL_n(\mathbb{Z}), k \otimes \mathbf{S}_G) \simeq H^1(SL_n(\mathbb{Z}), k).$$

This last group is zero by (3.12) or [K]. $\blacksquare$

The duality theorem and (2.3) yield a partial description of the top (co)homology of Γ in terms of representations of $SL_n(\mathbb{Z})$. This generalizes a result for trivial coefficients, obtained by Ash in [A1] using topological methods.

Let Γ be a subgroup of finite index in $SL_n(\mathbb{Z})$, A a right Γ module for which (3.1) holds (see (3.2)). Define

$$J(\Gamma, A) = \operatorname{Hom}_\Gamma(SL_n(\mathbb{Z}), A).$$

We will show that $H^v(\Gamma, A)$ (resp. $H_v(\Gamma, A)$) is a quotient (resp. subspace) of $J(\Gamma, A)$. Set

$$N_T = \operatorname{Norm}_{SL_n(\mathbb{Z})}(T).$$

For $\alpha \in \Delta$, let g_α be as in the proof of (3.6).

Let λ denote the left action of $SL_n(\mathbb{Z})$ on $J(\Gamma, A)$ given by

$$[\lambda(g)f](x) = f(g^{-1}x).$$

Set

$$J(\Gamma, A)_\varepsilon = \langle(\varepsilon(n) - \lambda(n))f \mid n \in N_T, f \in J(\Gamma, A)\rangle,$$

and

$$J(\Gamma, A)^1 = \{f \in J(\Gamma, A) \mid \lambda(g_{\alpha_1})f = f\}.$$

Finally, we define

$$K(\Gamma, A) = J(\Gamma, A)/J(\Gamma, A)_\varepsilon + J(\Gamma, A)^1,$$

and

$$W(\Gamma, A) = \{f \in J(\Gamma, A) \mid \lambda(n)f = \varepsilon(n)f \, \forall n \in N_T, \text{ and } (1 + \lambda(g_\alpha) + \lambda(g_\alpha^2))f = 0\}.$$

If Γ is normal in $SL_n(\mathbb{Z})$, then $J(\Gamma, A)$ is also a left $SL_n(\mathbb{Z})/\Gamma$ module via $(g \cdot f)(x) = [f(xg)] \, g^{-1}$. This action lifts to $K(\Gamma, A)$ and preserves $W(\Gamma, A)$.

(3.9) Proposition. *There is a surjection*

$$m^\Gamma: K(\Gamma, A) \twoheadrightarrow H^\nu(\Gamma, A)$$

and an injection

$$m_\Gamma: H_\nu(\Gamma, A) \hookrightarrow W(\Gamma, A).$$

These are $SL_n(\mathbb{Z})$-equivariant if Γ is normal in $SL_n(\mathbb{Z})$.

Proof. Define $m^\Gamma: J(\Gamma, A) \to \mathrm{S}_G \otimes_\Gamma A$ by

$$m^\Gamma(f) = \sum_{g \in SL_n(\mathbb{Z})/\Gamma} \tau \cdot g \otimes f(g).$$

It follows from our previous remarks that m^Γ lifts to $K(\Gamma, A)$. Define elements of $J(\Gamma, A)$ as follows: for $a \in A$, $g \in SL_n(\mathbb{Z})$, set

$$f_{g,a}(x) = a \cdot \gamma \text{ if } x = g\gamma, \qquad f_{g,a}(x) = 0 \text{ if } x \notin g\Gamma.$$

Note that

$$\sum_{g \in SL_n(\mathbb{Z})/\Gamma} \tau \cdot x \otimes f_{g,a}(x) = \tau \cdot g \otimes a.$$

It follows from (2.3) that the $\tau \cdot g \otimes a$ generate $\mathbf{S}_G \otimes_\Gamma A$. This shows surjectivity. The $SL_n(\mathbb{Z})/\Gamma$ equivariance is clear.

Finally, for $\varphi \in \mathrm{Hom}_\Gamma(\mathbf{S}_G, A)$ define $m_\Gamma(\varphi)$: $SL_n(\mathbb{Z}) \to A$ by $m_\Gamma(\varphi(g)) = \varphi(\tau \cdot g)$. It is easy to check that $m_\Gamma(\varphi) \in W(\Gamma, A)$. Using (2.3) again, we see that m_Γ is injective. ∎

(3.10) COROLLARY. *Let A be a $SL_n(\mathbb{Z})$ module with no ε-component for $N_T(\mathbb{Z})$. Then $H_\nu(SL_n(\mathbb{Z}), A)$ is all torsion. Moreover, if $H_r(SL_n(\mathbb{Z}), A)$ has p-torsion for $r = \nu$ or $\nu - 1$ then $p \leqslant 1 + n$. In particular,*

$$H_\nu(SL_n(\mathbb{Z}), \mathbb{Q}) = 0,$$

and

$$H_\nu(SL_n(\mathbb{Z}), \mathfrak{g}) = 0,$$

where $\mathfrak{g}$ is the Lie algebra of $SL_n(\mathbb{C})$ and $SL_n(\mathbb{Z})$ acts via the adjoint representation.

Proof. Let $p > 1 + n$. If $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F}_p$ then (3.9) applies to $\Gamma = SL_n(\mathbb{Z})$ so $H_\nu(SL_n(\mathbb{Z}), A \otimes \mathbb{F}) \subseteq W(SL_n(\mathbb{Z}), A \otimes \mathbb{F}) = 0$. The first statement now follows from the universal coefficient theorem. For the adjoint representation, we note that the Lie algebra of $T(\mathbb{C})$ affords the reflection representation of W, and does not contain the sign representation ε. Also, if $\mathbf{G}$ is not of type A_2, then every root is orthogonal to another root. This is clear if the rank is at least 4 and the rest are easily checked case by case. It follows that

$$\sum_{w \in W} \varepsilon(w) X \cdot \mathrm{Ad}(w) = 0$$

for every $X \in \mathfrak{g}^{T(\mathbb{Z})}$. The above equation is easy to verify when $\mathfrak{g} \simeq \mathfrak{sl}_3$ as well, so $\mathfrak{g}$ always has zero ε-component.

We can generalize (3.10) with a proof that is more specific to SL_n. If T is the diagonal torus, then $N_T(\mathbb{Z}) = SO_n(\mathbb{Z})$, the group of integral orthogonal matrices with determinant one. Let E be the natural representation of $SL_n(\mathbb{C})$ on $\mathbb{C}^n$.

(3.11) COROLLARY. *Let V be any constituent of $\Lambda^*(E \oplus E^*)$, where Λ^* denotes exterior algebra. Then*

$$H_\nu(SL_n(\mathbb{Z}), V) = 0.$$

Proof. Note that $O_n(\mathbb{Z})$ is the Weyl group of type C_n, and $SO_n(\mathbb{Z})$ is the kernel of its sign representation σ. Moreover, E is the reflection representation of $O_n(\mathbb{Z})$. Let ρ be an extension of ε to $O_n(\mathbb{Z})$.

Recall that for any Weyl group, the exterior powers of its reflection representation are distinct and irreducible. Also, every complex representation of a Weyl group is self-dual. Hence, by Frobenius reciprocity,

$$\begin{aligned}
&\mathrm{Hom}_{SO_n(\mathbb{Z})}(\Lambda^i E \otimes \Lambda^j E, \varepsilon) \\
&\quad \simeq \mathrm{Hom}_{O_n(\mathbb{Z})}(\Lambda^i E \otimes \Lambda^j E, \rho \oplus \sigma\rho) \\
&\quad \simeq \mathrm{Hom}_{O_n(\mathbb{Z})}(\Lambda^i E, \Lambda^j E \otimes \rho) \oplus \mathrm{Hom}_{O_n(\mathbb{Z})}(\Lambda^i E, \Lambda^j E \otimes \sigma\rho) = 0,
\end{aligned}$$

since the four characters 1, σ, ρ, $\rho\sigma$ are distinct. ∎

Using invariant theory, it is also not hard to show that there is no ε-component in $S^r E$ ($S^* E$ is the symmetric algebra on E), where r is restricted as follows: n even or r odd $\Rightarrow r \leqslant n+1$, n odd and r even $\Rightarrow r \leqslant n(n-1)$. Hence for these r,

$$H_v(SL_n(\mathbb{Z})), S^r E) = 0.$$

We now return to more general $\mathbb{Q}$-groups. As mentioned in the Introduction, the following result was provided by Steinberg, and is a vast improvement over a previous estimate in the spirit of (3.10).

(3.12) THEOREM. *Let* $\mathbf{G}$ *be a Chevalley group with irreducible root system.*

(1) *If* $\mathbf{G}$ *is not of type* A_1, B_2, *or* G_2, *then* $H_1(G(\mathbb{Z}), \mathbb{Z}) \simeq F/2F$, *where* F *is the fundamental group of* $\mathbf{G}$.

(2) *If* $\mathbf{G}$ *has type* G_2, *then* $H_1(G(\mathbb{Z}), \mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$.

(3) *If* $\mathbf{G}$ *has type* B_2, *then* $H_1(G(\mathbb{Z}), \mathbb{Z}) \simeq F \oplus \mathbb{Z}/2\mathbb{Z}$. (*Note that* F *is either trivial or of order* 2.)

(4) *If* $\mathbf{G}$ *has type* A_1, *then* $H_1(G(\mathbb{Z}), \mathbb{Z}) \simeq \mathbb{Z}/12\mathbb{Z}$ *or* $(\mathbb{Z}/2\mathbb{Z})^2$, *according to whether* $\mathbf{G}$ *is* $\mathbf{SL}_2$ *or* $\mathbf{PSL}_2$.

Proof. First suppose $\mathbf{G}$ is simply connected. Then [S, Lemma 49] $G(\mathbb{Z})$ is generated by the $x_\alpha(1)$'s for $\alpha \in \Phi$. If we can find two roots α and β which generate a subsystem of type A_2, then

$$[x_\alpha(1), x_\beta(1)] = x_{\alpha+\beta}(\pm 1).$$

Conjugating by Weyl group representatives in $G(\mathbb{Z})$, we get $x_\gamma(1) \in DG(\mathbb{Z})$ (D means derived group) for all roots γ with the same length as α and β. In particular, this proves $G(\mathbb{Z}) = DG(\mathbb{Z})$ in the simply laced cases.

In the remaining cases of (1), we can find a rank 3 subsystem Ψ of type B_3 or C_3. Suppose it is B_3, and choose a base $\{\alpha, \beta, \gamma\}$ of Ψ with γ short and orthogonal to α. We have

$$[x_\beta(1), x_\gamma(1)] = x_{\beta+\gamma}(\pm 1)\, x_{\beta+2\gamma}(\pm 1).$$

Now α and β are long roots generating an A_2 subsystem so $x_{\beta+2\gamma}(1) \in DG(\mathbb{Z})$. Thus all short root $x_\delta(1)$'s lie in $DG(\mathbb{Z})$ as well. If Ψ has type C_3, just interchange "long" and "short" in the above.

Now suppose $\mathbf{G}$ is not simply connected and let $\pi\colon \hat{\mathbf{G}} \to \mathbf{G}$ be the universal cover of $\mathbf{G}$. Let $\hat{\mathbf{T}}$ be a maximal split torus in $\hat{\mathbf{G}}$, and set $\mathbf{T} = \pi(\hat{\mathbf{T}})$. Lemma 49 of [S] now says that $G(\mathbb{Z})$ is generated by $T(\mathbb{Z})$ and the $x_\alpha(1)$'s. This gives a surjection

$$T(\mathbb{Z})/\pi\hat{T}(\mathbb{Z}) \to G(\mathbb{Z})/\pi\hat{G}(\mathbb{Z}). \tag{$*$}$$

In particular, $DG(\mathbb{Z}) \subseteq \pi\hat{G}(\mathbb{Z})$. From the simply connected case, we get $DG(\mathbb{Z}) = \pi\hat{G}(\mathbb{Z})$. We claim that $(*)$ is an isomorphism. Let $t \in T(\mathbb{Z}) \cap \pi\hat{G}(\mathbb{Z})$, and choose $x \in \hat{G}(\mathbb{Z})$, $h \in \hat{T}(\mathbb{C})$ such that $\pi(x) = t = \pi(h)$. Then $x = fh$ for some $f \in F \subset \hat{T}(\mathbb{C})$. Thus $x \in \hat{G}(\mathbb{Z}) \cap \hat{T}(\mathbb{C}) = \hat{T}(\mathbb{Z})$.

The isomorphism

$$T(\mathbb{Z})/\pi\hat{T}(\mathbb{Z}) \simeq F/2F$$

can be seen as follows. First note that $T(\mathbb{Z})$ and $\hat{T}(\mathbb{Z})$ consist of the involutions in $T(\mathbb{C})$ and $\hat{T}(\mathbb{C})$, respectively. If $t \in T(\mathbb{Z})$, let $\hat{t}$ be any lift in $\hat{T}(\mathbb{C})$. Then $\hat{t}^2 \in F$ and is well-defined modulo $2F$. For the inverse, note that $F \subset \hat{T}(\mathbb{C}) \simeq \mathbb{C}^* \times \cdots \times \mathbb{C}^*$, so every $f \in F$ has a square root $\sqrt{f}$, defined up to an involution in $\hat{T}(\mathbb{C})$, and $\pi(\sqrt{f})$ is an involution in $T(\mathbb{C})$. This completes the proof of (1).

For G_2 and B_2, let α and β be simple roots with α short. First consider G_2. The long root subsystem has type A_2, so $x_\gamma(1) \in DG(\mathbb{Z})$ for long roots γ. Also, $[x_\alpha(1), x_\beta(1)] = x_{2\alpha+\beta}(\pm 2) \bmod DG(\mathbb{Z})$. The Weyl group acts transitively on the short roots and trivially on $G(\mathbb{Z})/DG(\mathbb{Z})$, so $G(\mathbb{Z})/DG(\mathbb{Z})$ is generated by $x_\alpha(1)$ and has order one or two. However, it is known (cf. [Co]) that the abelianization of $G_2(\mathbb{Z}/2\mathbb{Z})$ has order two, so the same is true of $G_2(\mathbb{Z})$.

For B_2 we have the relations

$$[x_\alpha(1), x_\beta(1)] = x_{\alpha+\beta}(\pm 1)\, x_{2\alpha+\beta}(\pm 1)$$

$$[x_\alpha(1), x_{\alpha+\beta}(1)] = x_{2\alpha+\beta}(\pm 2),$$

which imply that $x_{\alpha+\beta}(1) = x_{2\alpha+\beta}(1) \bmod DG(\mathbb{Z})$. Conjugating this equation by the Weyl group, we get $G(\mathbb{Z})/DG(\mathbb{Z})$ either trivial or $\mathbb{Z}/2\mathbb{Z}$, in the simply connected case.

On the other hand, $SP_4(\mathbb{Z}/2\mathbb{Z}) \simeq S_6$ (symmetric group) as can be seen by reducing the reflection representation of the Weyl group of type A_5 modulo two (cf. [S]). It follows that the derived group of $SP_4(\mathbb{Z})$ is the kernel of reduction mod two followed by the sign character of S_6, hence has index two. We may take the image of any $x_\gamma(1)$ as a generator of the abelianization.

If $\mathbf{G}$ is of adjoint type, the fundamental group F is of order two, generated by $h_\alpha(-1)$. Since $h_\alpha(-1)$ is in the kernel of reduction mod 2, we have $F \subset DG(\mathbb{Z})$, so $\pi G(\mathbb{Z})/\pi DG(\mathbb{Z})$ has order two. As in the rank $\geqslant 3$ case, we have $G(\mathbb{Z})/\pi\hat{G}(\mathbb{Z}) \simeq F/2F$. Hence $[G(\mathbb{Z}) : \pi\, D\hat{G}(\mathbb{Z})] = 4$. In particular, $DG(\mathbb{Z}) \subseteq \pi\, D\hat{G}(\mathbb{Z})$. The other containment is clear, so $[\mathrm{G}(\mathbb{Z}) : DG(\mathbb{Z})] = 4$. Note that $h_\alpha(\sqrt{-1})$ is not contained in $\pi\hat{G}(\mathbb{Z})$, so its image in $G(\mathbb{Z})/DG(\mathbb{Z})$ is another involution, besides an $x_\gamma(1)$ from $\pi\hat{G}(\mathbb{Z})/\pi\, D\hat{G}(\mathbb{Z})$. It follows that $G(\mathbb{Z})/DG(\mathbb{Z}) \simeq (\mathbb{Z}/2\mathbb{Z})^2$.

Finally, suppose $\mathbf{G}$ has type A_1 and α is the positive root. The group $SL_2(\mathbb{Z})$ made abelian is well known to be cyclic of order 12, generated by the image of $x = x_\alpha(1)$. Moreover, $-I$ is not in the derived group. Let $\mathbf{G}$ be the adjoint group, so that $\hat{\mathbf{G}} = \mathbf{SL}_2$. We have $\pi\hat{G}(\mathbb{Z})/\pi\, D\hat{G}(\mathbb{Z}) \simeq \mathbb{Z}/6\mathbb{Z}$, generated by x, and $G(\mathbb{Z})/\pi\hat{G}(\mathbb{Z}) \simeq F \simeq \mathbb{Z}/2\mathbb{Z}$, generated by $h = h_\alpha(\sqrt{-1})$. Since $hxh^{-1} = x^{-1}$, we see that $G(\mathbb{Z})/\pi\, D\hat{G}(\mathbb{Z})$ is dihedral or order 12. It follows that $G(\mathbb{Z})/DG(\mathbb{Z}) \simeq (\mathbb{Z}/2\mathbb{Z})^2$.

Remark. In [B1], it is pointed out that $T(\mathbb{Z})/\pi\hat{T}(\mathbb{Z})$ is also isomorphic to the center of $\hat{G}(\mathbb{Z})$.

4. The Restriction Map

Retain the set-up of Section 3, except now $\mathbf{G}$ need not be split. Let $\mathbf{P}$ be a parabolic $\mathbb{Q}$-subgroup of $\mathbf{G}$. We consider the restriction of cohomology from Γ to $\Gamma \cap P$. Now $\Gamma \cap P$ is also a duality group with dualizing module $\mathbf{S}_P$, and the same dualizing dimension v.

We find the missing link in the commutative diagram

$$\begin{tikzcd} H^q(\Gamma, A) \arrow[r] \arrow[d, "\mathrm{res}"] & H_{v-q}(\Gamma, \mathbf{S}_G \otimes A) \arrow[d, "?"] \\ H^q(\Gamma \cap P, A) \arrow[r] & H_{v-q}(\Gamma \cap P, \mathbf{S}_P \otimes A). \end{tikzcd}$$

We then let $q = v$ and give a more explicit formula for the right hand map.

We will define a "transfer" map

$$H_*(\Gamma, \mathbf{S}_G \otimes A) \to H_*(\Gamma \cap P, \mathbf{S}_P \otimes A)$$

even though $[\Gamma : \Gamma \cap P] = \infty$. This is possible because $\mathbf{S}_G$ and $\mathbf{S}_P$ are contained in modules induced from a subgroup of P.

Let us temporarily suppose $B < P < G$ is any containment of groups and S is a G submodule of $\mathbb{Z} \otimes_B G$. Decompose $\mathbb{Z} \otimes_B G$ as a P module to get

$$\mathbb{Z} \otimes_B G = \bigoplus_{g \in B\backslash G/P} \mathbb{Z} \otimes_B BgP.$$

Let s_P be the projection on to the summand for $g = 1$ and let $S_P = s_P(S)$.

(4.1) LEMMA. *Let Γ be a subgroup of G, $\eta \in S$. Then the map $\Gamma \to S_P$ given by $\gamma \mapsto s_P(\eta \cdot \gamma)$ is supported on finitely many cosets*

$$\gamma(\Gamma \cap P) \qquad \textit{for} \quad \gamma \in \Gamma.$$

Proof. It is enough to prove this with S and S_P replaced by $\mathbb{Z} \otimes_B G$ and $\mathbb{Z} \otimes_B P$, respectively. Then we may assume $\eta = 1 \otimes g$. If $g\Gamma \cap P = \varnothing$ then $s_P(\eta \cdot \gamma)$ is always zero. Hence we suppose there is a $\gamma_g \in g^{-1}P$. Now for any $\gamma \in \Gamma$, we have $s_P(1 \otimes g\gamma) \neq 0 \Leftrightarrow g\gamma \in P$. It follows that $\gamma\gamma_g^{-1} \in \Gamma \cap P$. ∎

Let $\Gamma < G$, A a right $\mathbb{Z}\Gamma$ module. To ease the notation, we set $\tilde{A} = S \otimes A$, $\tilde{A}_P = S_P \otimes A$, $\rho = s_P \otimes 1_A \in \mathrm{Hom}_{\Gamma \cap P}(\tilde{A}, \tilde{A}_P)$. Let $C. \to \mathbb{Z}$ be a projective resolution of $\mathbb{Z}$ of right $\mathbb{Z}G$ modules. Define

$$\Theta_{A,P}\colon C_n \otimes_\Gamma \tilde{A} \to C_n \otimes_{\Gamma \cap P} \tilde{A}_P$$

by

$$\Theta_{A,P}(x \otimes \tilde{a}) = \sum_{\gamma \in \Gamma/\Gamma \cap P} x \cdot \gamma \otimes \rho(\tilde{a} \cdot \gamma).$$

The sum makes sense by (4.1), and $\Theta_{A,P}$ is clearly a chain map, so we get

$$\Theta_{A,P}\colon H_n(\Gamma, \tilde{A}) \to H_n(\Gamma \cap P, \tilde{A}_P).$$

If there is no confusion, we write Θ_A for $\Theta_{A,P}$. We remark for later use that if $n = 0$ and $H_0(\Gamma, \tilde{A})$, $H_0(\Gamma \cap P, \tilde{A}_P)$ are identified with $\mathbb{Z} \otimes_\Gamma \tilde{A}$ and $\mathbb{Z} \otimes_{\Gamma \cap P} \tilde{A}_P$, respectively, then $\Theta_A\colon \mathbb{Z} \otimes_\Gamma \tilde{A} \to \mathbb{Z} \otimes_{\Gamma \cap P} \tilde{A}_P$ is given by

$$\Theta_A(1 \otimes \tilde{a}) = \sum_{\gamma \in \Gamma/\Gamma \cap P} 1 \otimes \rho(\tilde{a} \cdot \gamma).$$

(4.2) LEMMA. *The following diagram commutes*:

$$\begin{array}{ccc}
\cap : H_q(\Gamma, \mathbf{S}) \otimes H^p(\Gamma, A) & \longrightarrow & H_{q-p}(\Gamma, \mathbf{S} \otimes A) \\
\Big\downarrow {\scriptstyle \Theta_{\mathbb{Z}} \otimes \mathrm{res}} & & \Big\downarrow {\scriptstyle \Theta_{A,P}} \\
\cap : H_q(\Gamma \cap P, \mathbf{S}_P) \otimes H^p(\Gamma \cap P, A) & \longrightarrow & H_{q-p}(\Gamma \cap P, \mathbf{S}_P \otimes A).
\end{array}$$

Proof. This is identical to the proof for the ordinary transfer map. See [Br]. ∎

Now let $\mathbf{G}$ be a $\mathbb{Q}$-group, Γ a torsion-free arithmetic subgroup with $\omega_\Gamma \equiv 1$, $\mathbf{B} \subseteq \mathbf{P}$ a containment of $\mathbb{Q}$-parabolics of $\mathbf{G}$ with $\mathbf{B}$ minimal.

Choose generators $e \in H_v(\Gamma, \mathbf{S}_G)$, $e_P \in H_v(\Gamma \cap P, \mathbf{S}_P)$ as in (3.1).

(4.3) Lemma. $\Theta_{\mathbb{Z}}: H_\nu(\Gamma, \mathbf{S}_G) \to H_\nu(\Gamma \cap P, \mathbf{S}_P)$ *is an isomorphism.*

Proof. We have $\Theta_{\mathbb{Z}}(e) = ke_P$ for some $k \in \mathbb{Z}$. We apply (4.2) and the remark prior to it to get a commutative diagram

$$\begin{array}{ccc} H_\nu(\Gamma, \mathbf{S}_G) \otimes H^\nu(\Gamma, \mathbb{Z}\Gamma) & \longrightarrow & \mathbf{S}_G \\ \downarrow{\scriptstyle \Theta_{\mathbb{Z}} \otimes \mathrm{res}} & & \downarrow{\scriptstyle \Theta_{\mathbb{Z}\Gamma}} \\ H_\nu(\Gamma \cap P, \mathbf{S}_P) \otimes H^\nu(\Gamma \cap P, \mathbb{Z}\Gamma) & \longrightarrow & \mathbf{S}_P \otimes_{\Gamma \cap P} \mathbb{Z}\Gamma. \end{array}$$

Note that

$$\Theta_{\mathbb{Z}\Gamma}(x) = \sum_{\gamma \in \Gamma/\Gamma \cap P} s_P(x \cdot \gamma) \otimes \gamma^{-1}.$$

For $\varphi \in H^\nu(\Gamma, \mathbb{Z}\Gamma)$, we have $\Theta_{\mathbb{Z}\Gamma}(e \cap \varphi) = k[e_P \cap \mathrm{res}(\varphi)]$. Now

$$\mathbf{S}_P \otimes_{\Gamma \cap P} \mathbb{Z}\Gamma \simeq (\mathbf{S}_P \otimes 1) \oplus \sum_{1 \neq \gamma \in \Gamma/\Gamma \cap P} \mathbf{S}_P \otimes \mathbb{Z}[\gamma^{-1}\Gamma \cap P],$$

as $\mathbb{Z}[\Gamma \cap P]$ modules. Let

$$\Pi: \mathbf{S}_P \otimes_{\Gamma \cap P} \mathbb{Z}\Gamma \to \mathbf{S}_P$$

denote the projection onto the first factor. Let $\tau \in \mathbf{S}_G$, $\tau_P \in \mathbf{S}_P$ correspond to the maximal $\mathbb{Q}$-split torus $T < P$. Then

$$\Pi[\Theta_{\mathbb{Z}\Gamma}(\tau)] = \Pi\left[\tau_P \otimes 1 + \sum_{1 \neq \gamma \in \Gamma/\Gamma \cap P} s_P(\nu\gamma) \otimes \gamma^{-1}\right] = \tau_P.$$

Now let $\varphi \in H^\nu(\Gamma, \mathbb{Z}\Gamma)$ be such that $e \cap \varphi = \tau$. Then

$$\tau_P = \Pi(\Theta_{\mathbb{Z}\Gamma}(\tau)) = \Pi(k(e_P \cap \mathrm{res}(\varphi))) \in k\mathbf{S}_P.$$

But $k\mathbf{S}_P$ is P-invariant and τ_P generates $\mathbf{S}_P$ over P, so we must have $k\mathbf{S}_P = \mathbf{S}_P$. Since $\mathbf{S}_P$ is free abelian, we see that $k = \pm 1$. The lemma follows. ∎

(4.4) Proposition. *We may choose e, e_P so that for every Γ-module A there is a commutative diagram*

$$\begin{array}{ccc} H^q(\Gamma, A) & \xrightarrow[\simeq]{e \cap} & H_{\nu-q}(\Gamma, \mathbf{S}_G \otimes A) \\ \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \Theta_A} \\ H^q(\Gamma \cap P, A) & \xrightarrow[\simeq]{e_P \cap} & H_{\nu-q}(\Gamma \cap P, \mathbf{S}_P \otimes A). \end{array}$$

Proof. By (4.3), we may take

$$e_P = \Theta_{\mathbb{Z}}(e).$$

Theorem (3.1) says that $e\cap$ and $e_P\cap$ are isomorphisms. The proposition now follows from (4.2). ∎

(4.5) *Remark.* Assume $\operatorname{rank}_{\mathbb{Q}} \mathbf{G}$. When $\mathbf{P}=\mathbf{G}$, a minimal $\mathbb{Q}$-parabolic subgroup, the restriction map has an interesting interpretation as follows. Set

$$X=\mathbb{Z}\otimes_B G, \qquad X_\mu=\mathbb{Z}\otimes_{\mu\Gamma\mu^{-1}\cap B}\Gamma.$$

Then

$$X\simeq \bigotimes_{\mu\in B\backslash G/\Gamma} X_\mu, \qquad \text{as } \Gamma \text{ modules.}$$

The inclusion $\mathbf{S}_G\to X$ induces a map

$$H_*(\Gamma,\mathbf{S}_G\otimes A)\to H_*(\Gamma, X\otimes A).$$

Also,

$$\begin{aligned}
H_*(\Gamma, X\otimes A) &\simeq \bigoplus_\mu H_*(\Gamma, X_\mu\otimes A)\\
&\simeq \bigoplus_\mu H_*(\Gamma\cap B^\mu, A) \qquad \text{(Shapiro's lemma)}\\
&\simeq \bigoplus_\mu H^{\nu-*}(\Gamma\cap B^\mu, A) \qquad \text{(Poincaré Duality).}
\end{aligned}$$

Finally, we have restriction maps

$$H^{\nu-*}(\Gamma, A)\to \bigoplus_\mu H^{\nu-*}(\Gamma\cap B^\mu, A).$$

Using (4.4), it is not difficult to show that these maps fit into the commutative diagram

$$\begin{tikzcd}
H^{\nu-*}(\Gamma, A) \arrow[r, "e\cap"', "\simeq"] \arrow[d] & H_*(\Gamma,\mathbf{S}_G\otimes A) \arrow[d]\\
\bigoplus_\mu H^{\nu-*}(\Gamma\cap B^\mu, A) \arrow[r, "\simeq"'] & H_*(\Gamma, X\otimes A).
\end{tikzcd}$$

In particular, if we dualize this diagram, we see that the image of $\bigoplus_\mu H_\nu(\Gamma\cap B^\mu, A)$ in $H_\nu(\Gamma, A)$ consists precisely of those $\varphi\in\operatorname{Hom}_\Gamma(\mathbf{S}_G, A)$ which extend to X.

We can give a more explicit formula for the restriction map on the top cohomology groups. Fix a $\mathbb{Q}$-parabolic P and a maximal $\mathbb{Q}$-split torus T of a Levi subgroup L of P. Let τ and τ_P be the corresponding elements of

$\mathbf{S}_G$ and $\mathbf{S}_P$. We will describe the restriction of $\tau \cdot g \otimes a \in \mathbf{S}_G \otimes_\Gamma A$ to a conjugate $Q = P^\mu$ in terms of τ_P, μ, and P.

(4.6) *Notation.* Suppose $Q = \mu^{-1} P \mu$, for some $\mu \in G$, and A is in fact a module for the subgroup of G generated by Γ and μ. We have an isomorphism $\mu_P: \mathbf{S}_Q \otimes_{\Gamma \cap Q} A \to \mathbf{S}_P \otimes_{\mu\Gamma\mu^{-1} \cap P} A$ given by $x \otimes a \mapsto c_\mu x \otimes a\mu^{-1}$. (Remark (1.2) gives the definition of c_μ and is used to show this is well defined.) We also denote by $\mu_G: \mathbf{S}_G \otimes_\Gamma A \to \mathbf{S}_G \otimes_{\mu\Gamma\mu^{-1}} A$ the map $\mu_G(x \otimes a) = x\mu \otimes a\mu^{-1}$. It is easy to verify that

$$\mu_P \circ (s_Q \otimes 1_A) = (s_P \otimes 1_A) \circ \mu_G.$$

We set $\Gamma(P, \mu) := \mu\Gamma\mu^{-1} \cap P$, $\Gamma(L, \mu) := \pi(\mu\Gamma\mu^{-1} \cap P)$, $\Gamma(N, \mu) := \mu\Gamma\mu^{-1} \cap N$. We abbreviate

$$A_\mu := A_{\Gamma(N,\mu)} = \text{the coinvariants of } \Gamma(N, \mu) \text{ in } A,$$

and let $a \mapsto \bar{a}: A \to A_\mu$ be the canonical map. Since $\Gamma(N, \mu)$ acts trivially on $\mathbf{S}_P$, there is a canonical identification

$$\mathbf{S}_P \otimes_{\Gamma(P,\mu)} A = \mathbf{S}_P \otimes_{\Gamma(L,\mu)} A_\mu.$$

Finally, define a linear map

$$p_\mu: \mathbb{Z}G \to \mathbb{Z}[P/\Gamma(P, \mu)]$$

by

$$\begin{aligned} p_\mu(g) &= \text{the coset of } p \text{ if } g \in p\mu\Gamma \\ &= 0 \text{ if } g \notin P\mu\Gamma. \end{aligned}$$

(4.7) Proposition. *Let Γ be a torsion-free arithmetic subgroup of G, $\mathbf{P} = \mathbf{LN}$ (Levi decomposition) a $\mathbb{Q}$-parabolic of $\mathbf{G}$, $\mathbf{Q} = \mathbf{P}^\mu$, where $\mu \in G$. Let $\mathbf{T}$ be a maximal $\mathbb{Q}$-split torus of $\mathbf{L}$ with corresponding elements τ, τ_L in $\mathbf{S}_G$, $\mathbf{S}_P$. Define a map*

$$r(P, \mu, A): \mathbf{S}_G \otimes_\Gamma A \to \mathbf{S}_P \otimes_{\Gamma(L,\mu)} A_\mu$$

by

$$r(P, \mu, A) = \mu_P \circ \Theta_{A,Q} \qquad (\textit{see } (4.4) \textit{ and } (4.6)).$$

Assume that A is the restriction to Γ of a module for a subgroup $\Omega < G$ such that Ω contains Γ, μ and a complete set of representatives of W. Then

(1) *There is a commutative diagram*

$$\begin{tikzcd} H^\nu(\Gamma, A) \arrow[r, "e\cap", "\simeq"'] \arrow[d, "\mathrm{res}"] & \mathbf{S}_G \otimes_\Gamma A \arrow[d, "r(P,\mu,A)"] \\ H^\nu(\Gamma \cap Q, A) \arrow[r, "\simeq"] & \mathbf{S}_P \otimes_{\Gamma(L,\mu)} A_\mu. \end{tikzcd}$$

(2) *If* $g \in \Omega$, $a \in A$, *then*

$$r(P, \mu, A)[\tau \cdot g \otimes a] = \sum_{w \in W_P \backslash W} [\varepsilon(w)\, \tau_L \cdot p_\mu(wg) \otimes \overline{[a \cdot (wg)^{-1}]} \cdot p_\mu(wg)].$$

Proof. The bottom isomorphism is the composition

$$x \mapsto \mu_P(e_Q \cap x) \colon H^v(\Gamma \cap Q, A) \to \mathbf{S}_Q \otimes_{\Gamma \cap Q} A \to \mathbf{S}_P \otimes_{\Gamma(\mu, P)} A = \mathbf{S}_L \otimes_{\Gamma(L, \mu)} A_\mu.$$

From (4.4), we get the commutative diagram

$$\begin{array}{ccccc} H^v(\Gamma, A) & \xrightarrow[\simeq]{e\,\cap} & \mathbf{S}_G \otimes_\Gamma A & & \\ \Big\downarrow \text{res} & & \Big\downarrow \Theta_{A,Q} & & \\ H^v(\Gamma \cap Q, A) & \xrightarrow[\simeq]{} & \mathbf{S}_Q \otimes_{\Gamma \cap Q} A & \xrightarrow[\simeq]{\mu_P} & \mathbf{S}_L \otimes_{\Gamma(L,\mu)} A_\mu. \end{array}$$

Now (1) follows from the definition of $r(P, \mu, A)$.

For (2), we begin with a

LEMMA.

$$s_P(\tau \cdot g) = \begin{cases} \varepsilon(w)\, \tau_L \cdot wg & \text{if } wg \in P \text{ for some } w \in W \\ 0 & \text{if } g \notin WP. \end{cases}$$

Note that the right hand side does not depend on the representative of w since $T < P$ and the centralizer of T in G fixes τ_L.

Proof. The faces of the apartment $\tau \cdot g$ are the parabolics which contain $g^{-1}Tg$. If $s_p(\tau \cdot g) \neq 0$, then P is one of these, so $T < gPg^{-1}$. Choose minimal parabolics B_1 and B_2 contained in P and gPg^{-1}, respectively, with $T \subset B_1 \cap B_2$. Since W is transitive on the minimal parabolics containing T, $\exists w \in W$ such that $B_2 = w^{-1} B_1 w$. Since P is transitive on its own minimal parabolics, $\exists p \in P$ with $PB_1 p^{-1} = g^{-1} B_2 g = g^{-1} B_1 wg$. It follows that $wg \in P$.

Now assume $wg \in P$, for some w. By (1.1),

$$\tau_L \cdot wg = s_P(\tau) \cdot wg = s_P(\tau \cdot wg) = \varepsilon(w)\, s_P(\tau \cdot g).$$

This proves the lemma.

We now compute

$$\begin{aligned}
&\mu_P \circ \Theta_{A,Q}[\tau \cdot g \otimes a] \\
&\quad = \mu_P \left[\sum_{\gamma \in \Gamma/\Gamma \cap Q} s_Q(\tau \cdot g\gamma) \otimes a \cdot \gamma \right] \qquad \text{(by (4.2))} \\
&\quad = \sum_{\gamma \in \Gamma/\Gamma \cap Q} s_P(\tau \cdot g\gamma\mu^{-1}) \otimes \overline{(a \cdot \gamma\mu^{-1})} \qquad \text{(by (4.6))} \\
&\quad = \sum_{\gamma \in \mu\Gamma\mu^{-1}/\Gamma(P,\mu)} s_P(\tau \cdot g\mu^{-1}\gamma) \otimes \overline{(a \cdot \mu^{-1}\gamma)} \\
&\quad = \sum \varepsilon(w)\, \tau_L \cdot wg\mu^{-1}\gamma \otimes \overline{(a \cdot \mu^{-1}\gamma)},
\end{aligned}$$

where the sum is over $\{(\gamma, w) \in \mu\Gamma\mu^{-1}/\Gamma(P, \mu) \times W_P\backslash W \mid wg\mu^{-1}\gamma \in P\}$, by the lemma. It is easy to see that projection onto the second factor gives a bijection from this set onto $\{w \in W_P\backslash W \mid wg \in P\mu\Gamma\}$.

If $x \in P\mu\Gamma$, write $x = p_\mu(x)\, \mu\gamma(x)$ where $\gamma(x) \in \Gamma$. Two choices of $p_\mu(x)$ or $\gamma(x)$ differ by an element of $\Gamma(P, \mu)$. If $wg\mu^{-1}\gamma \in P$, with $\gamma \in \mu\Gamma\mu^{-1}$, then $wg \in P\mu\Gamma$ and we may take $p_\mu(wg) = wg\mu^{-1}\gamma$, $\gamma(wg) = \mu^{-1}\gamma^{-1}\mu$. Hence

$$\mu_P \circ \Theta_{A,Q}[\tau \cdot g \otimes a] = \sum \varepsilon(w)\, \tau_L \cdot p_\mu(wg) \otimes \overline{[a \cdot (wg)^{-1}]} \cdot p_\mu(wg),$$

where the sum is over $w \in W_P\backslash W$ such that $wg \in P\mu\Gamma$. Since p_μ is defined to be zero off $P\mu\Gamma$, we can just sum over all $W_P\backslash W$, and this is the statement of (2). ∎

(4.8) *Remarks.* (i) If A is a rational $\mathbb{C}G$ module then $A_{\Gamma(N,\mu)} = A_N$ since $\Gamma(N, \mu)$ is Zariski dense in $N(\mathbb{C})$. Hence we may, in the formula above, replace $p_\mu(wg)$ by its projection into $\mathbb{Z}[L/\Gamma(L, \mu)]$

(ii) If A is a $G(\mathbb{Q})$ module, this formula gives the value of $r(P, \mu, A)$ on all of $\mathbf{S}_G \otimes_\Gamma A$.

We will give the homology version of (4.7). Define

$$j\colon \operatorname{Hom}_{\Gamma(P,\mu)}(\mathbf{S}_P, A) \to \operatorname{Hom}_\Gamma(\mathbf{S}_G, A)$$

by

$$j(\varphi)(x) = \sum_{\mu\Gamma\mu^{-1}/\Gamma(P,\mu)} \varphi(s_P(x \cdot \mu^{-1}\gamma)) \cdot \gamma^{-1}\mu.$$

Let

$$i_*\colon H_v(\Gamma(P, \mu), A) \to H_v(\Gamma, A)$$

be the homology map induced by the inclusion $\Gamma(P, \mu) < \mu\Gamma\mu^{-1}$, followed by the natural map $H_\nu(\mu\Gamma\mu^{-1}, A) \to H_\nu(\Gamma, A)$.

(4.9) PROPOSITION. (1) *The maps i_* and j correspond under Borel–Serre duality.*

(2) *For $g \in \Omega$, $\varphi \in \mathrm{Hom}_{\Gamma(L,\mu)}(\mathbf{S}_L, A^{\Gamma(N,\mu)})$, we have*

$$j(\varphi)(\tau \cdot g) = \sum_{w \in W_P \backslash W} \varepsilon(w)\, \varphi[\tau_L \cdot p_\mu(wg)] \cdot p_\mu(wg)^{-1}\, wg.$$

(*Here $p_\mu(wg)^{-1}$ is to be read as zero if $wg \notin P\mu\Gamma$*).

Proof. This is an easy consequence of (4.7).

(4.10) *Remark.* If char(k) is zero or sufficiently large, then Borel–Serre duality holds when $[\mathbf{G} = \mathbf{SL}_n, \Gamma = SL_n(\mathbb{Z})]$ and A is a $k\Gamma$ module (see (3.2)). In this case, we may assume $\mu = 1$ and $p_\mu(wg) \equiv 1$, so that

$$j(\varphi)(\tau \cdot g) = \sum_{w \in W_P \backslash W} \varepsilon(w)\, \varphi(\tau_L) \cdot wg.$$

Also, by (2.3), we have injections

$$\mathrm{Hom}_\Gamma(\mathbf{S}_G, A) \hookrightarrow A, \qquad \mathrm{Hom}_{\Gamma(L)}(\mathbf{S}_L, A^{\Gamma(N)}) \hookrightarrow A^{\Gamma(N)}$$

given by evaluation at τ and τ_L, respectively. We thus have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{G(\mathbb{Z})}(\mathbf{S}_G, A) \arrow[r] & A^\varepsilon \\ \mathrm{Hom}_{L(\mathbb{Z})}(\mathbf{S}_L, A^{\Gamma(N)}) \arrow[r] \arrow[u, "j"] & (A^{\Gamma(N)})^\varepsilon, \arrow[u, "\zeta"'] \end{tikzcd}$$

where $\zeta(a) = \sum_{w \in W_P \backslash W} \varepsilon(w) a \cdot w$, and the superscript ε denotes the ε-isotypic component.

Actually, since $L(\mathbb{R})$ may not be connected, even if $G(\mathbb{R})$ is, it might seem necessary to write $\mathbf{S}_L^\omega$, where ω is the orientation character of L on the symmetric space for $L(\mathbb{R})$. However, we are using Poincaré duality on $L(N)$ and it is easy to see that ω is also the orientation character of L on $N(\mathbb{R})$, so the ω's cancel.

(4.11) COROLLARY. *If there is a reflection in W which fixes $A^{\Gamma(N)}$ pointwise then the homology inclusion $H_\nu(P(\mathbb{Z}), A) \to H_\nu(SL_n(\mathbb{Z}), A)$ is identically zero.*

Proof. The hypothesis implies that ζ is identically zero. ∎

We now consider the surjectivity of the restriction map

$$\mathrm{res}\colon H^{\nu}(\Gamma, A) \to H^{\nu}(\Gamma \cap P, A).$$

We will define a map in the other direction which, for certain A, is a section to res. To do this, we use the section σ_L of s_P given in (1.1). We do not assume $\mathbf{G}$ is split over $\mathbf{Q}$.

Let Γ be a normal subgroup of $G(\mathbb{Z})$ having finite index (not necessarily torsion-free). Fix a minimal $\mathbb{Q}$-parabolic subgroup $\mathbf{B}$, a $\mathbb{Q}$-parabolic $\mathbf{P} = \mathbf{LN} \supseteq \mathbf{B}$ and let $\mathbf{T}$ be a maximal $\mathbb{Q}$-split torus of $\mathbf{L}$.

Let Δ (resp. Δ_L) be the simple $\mathbb{Q}$-roots of $\mathbf{T}$ in $\mathbf{G}$ (resp.) L determined by $\mathbf{B}$. Likewise, let Φ^+ and Φ_L^+ be the corresponding positive roots. For $\beta \in \mathbb{Z}\Delta$, we can uniquely write

$$\beta = \beta_L + \beta^L$$

with $\beta_L \in \mathbb{Z}^+\Delta_L$ and $\beta^L \in \mathbb{Z}^+[\Delta \backslash \Delta_L]$.

Let $(\,,\,)$ be a positive definite W-invariant inner product on $\mathbb{Q}\Delta$, and define $\langle \alpha, \beta \rangle = 2(\alpha, \beta)/(\beta, \beta)$.

Let V be an irreducible finite dimensional $G(\mathbb{C})$ representation with lowest weight $\tilde{\lambda}$, and let λ be the restriction of $\tilde{\lambda}$ to $T(\mathbb{C})$. We have an $L(\mathbb{C})$ isomorphism

$$V_N \simeq \bigoplus V_{\lambda+\nu},$$

where ν runs over $\mathbb{Z}^+\Delta_L$ and V_δ denotes the weight space for the weight δ.

Now $\Gamma \trianglelefteq G(\mathbb{Z}) \Rightarrow \Gamma(L) < \Gamma$; so we have a map

$$\sigma(L, V)\colon \mathbf{S}_L \otimes_{\Gamma(L)} V_N \to \mathbf{S}_G \otimes_\Gamma V$$

given by

$$\sigma(L, V)(x \otimes v) = \sigma_L(x) \otimes v.$$

The composition

$$r(P, V) \circ \sigma(L, V)\colon \mathbf{S}_L \otimes_{\Gamma(L)} V_N \to \mathbf{S}_L \otimes_{\Gamma(L)} V_N$$

is easily seen to be

$$\tau_L \cdot l \otimes v \mapsto \sum_{w \in W_P/W} [\varepsilon(w)\, \tau_L \cdot p(wl) \otimes \overline{(v \cdot l^{-1}w)} \cdot p(wl)],$$

where $l \in L$ and the bar denotes projection into V_N.

(4.12) THEOREM. *If $\langle \lambda, \alpha \rangle < 0$ for all $\alpha \in \Delta \backslash \Delta_L$ then the restriction map*

$$H^{\nu}(\Gamma, V) \to H^{\nu}(\Gamma \cap P, V)$$

is surjective.

Proof. We will show that under the hypotheses,

$$\overline{(V_N \cdot w)} \neq 0 \Rightarrow w \in W_P.$$

It will follow that $r(P, V) \circ \sigma(P, V)$ is the identity map on $S_L \otimes_{\Gamma(L)} V$. We need a lemma.

(4.13) LEMMA. *Let $y \in W$ and suppose η and $y\eta$ are weights of $T(\mathbb{C})$ in V_N with*

$$\langle \eta, \alpha \rangle \leqslant 0 \qquad \textit{for} \quad \alpha \in \Delta_L,$$

$$\langle \eta, \alpha \rangle < 0 \qquad \textit{for} \quad \alpha \in \Delta \backslash \Delta_L.$$

Then $y \in W_P$.

Proof. For any weight $\delta = \lambda + \sum_{\beta \in \Delta} c_\beta \beta$, set $\delta_\beta = c_\beta$. Let $\alpha \in \Delta \backslash \Delta_L$, $w' \in W$ be such that $l(s_\alpha w') = l(w') + 1$. Then one easily checks that

$$(s_\alpha w'\eta)_\alpha \geqslant (w'\eta)_\alpha.$$

Moreover, if $w' \in W_P$, this inequality is strict. Also, if $w' \in W_P$ then

$$(w'\eta)_\alpha = \eta_\alpha.$$

It follows that if s_α appears in a reduced expression for y then

$$(y\eta)_\alpha > \eta_\alpha.$$

But any two weights in V_N differ by an element of $\mathbb{Z}\Delta_L$, so $y\eta - \eta \in \mathbb{Z}\Delta_L$. Therefore s_α does not appear in a reduced expression for y, so $y \in W_P$. ∎

Proof of (4.12). Let v be a weight in V_N and write $v = \lambda + \gamma$, $\gamma \in \mathbb{Z}^+ \Delta_L$. If $\beta \in \Delta \backslash \Delta_L$ we have $\langle \gamma, \beta \rangle \leqslant 0$, so $\langle v, \beta \rangle > 0$. Suppose wv is also a weight in V_N. There is a $w_1 \in W_P$ such that $\langle w_1 v, \alpha \rangle \leqslant 0$ for all $\alpha \in \Delta_L$. In (4.13), set $\eta = w_1 v$, $y = ww_1^{-1}$. If $\beta \in \Delta \backslash \Delta_L$ then $\langle \eta, \beta \rangle = \langle v, w_1^{-1}\beta \rangle < 0$ since $w_1^{-1}\beta \in \Phi \backslash \Phi_L$. The theorem now follows from (4.13). ∎

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